Introduction: coming attractions

Monday, August 14, 2023 9:32 AM

Fundamental theorem of Algebra

On the invention of Complex numbers:

Look at a cubic equation of the form $t^3+pt+q=0$
Famous Cardano's formula(1545!):

$$t=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}.$$

Only one root? There are always three complex cubic roots!
$t^3-3t=0.\ p=-3,\ q=0.$ Nice real roots: 0, $\pm\sqrt{3}$
Formula:
t=$\sqrt[3]{\sqrt{-1}}+\sqrt[3]{\sqrt{-1}}$.
What is wrong? We need to take all possible values of cubic root of *i*!

Every (non-constant) polynomial with complex coefficients has a root.
In fact, if it has degree *d*, it has at least *d* roots, counting multiplicity.

Gerolamo Cardano

Complex differentiation vs real differentiation

Real case:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

- Function can be differentiable only once: $f(x)=\begin{cases}x^2, & x\geq 0\\ -x^2, & x\leq 0\end{cases}$

- Even infinitely differentiable functions can have nothing to do with their Taylor series: $\boxed{f(x)=\begin{cases}e^{-1/x^2}, & x\neq 0\\ 0, & x=0.\end{cases}\quad f^{(k)}(0)=0\ \forall k}$

$f(x)=\sum a_k(x-x_0)^k$ - real analytic functions.

Differentiable $\neq$ Infinitely Differentiable

Complex case:

$$f'(z)=\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$$ - analytic functions.

- Any function differentiable at a neighborhood of a point is infinitely differentiable.

- Differentiable functions are automatically equal to the sum of their Taylor series:

$$f(z)=\sum_{k=0}^{\infty}a_k(z-z_0)^k,\ a_k=\frac{f^{(k)}(z_0)}{k!}$$

Differentiable = Infinitely Differentiable = Analytic

Differentiable $\neq$ Infinitely Differentiable $\neq$ Analytic

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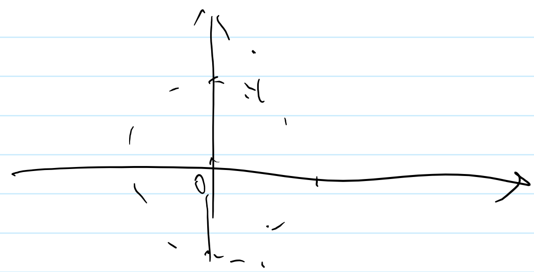
$\frac{1}{1-x} = \sum_{n=0} x^n$, $|x|<1$ $x=1$ singularity

$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$, $|x|<1$ No real singularities, but $z=\pm i$ - complex singularity



Complex Antiderivative

Real case:

Any continuous function has an antiderivative: $F'(x)=f(x)$

$F(x) := \int_0^x f(t)\,dt$ - FTC!

x $x+h$

Complex case:

Only differentiable functions have antiderivatives.

$F(z) := \oint_{z_0}^{z} f(w)\,dw$ - line integral.

But over which path?

$\oint_\gamma f(z)\,dz = 0$

$\left|\frac{f(z+h)-f(z)-T(h)}{h}\right| \to 0$

$\left|\frac{f(z+h)-f(z)-\lambda h}{h}\right| \to 0$, $\lambda = f'(z)$

Conformal maps and Riemann Theorem

Special important class: conformal maps: injective analytic functions.

$\varphi: \Omega_1 \to \Omega_2$ – angle and orientation preserving map.

Theorem (Riemann) If Ω is a simply-connected domain ("a domain without holes") $\Omega \neq \mathbb{C}$. Then $\exists\, \varphi: \mathbb{D} \to \Omega$ – conformal bijection!

Bernhard Riemann